Open/Closed Correspondence and Mirror Symmetry

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Open/Closed Correspondence

- Proposed in physics 20+ years ago as a class of string dualities
  - [Mayr '01, Lerche-Mayr '01]

open strings on Calabi-Yau 3-folds	$\xleftrightarrow[\text{topological amplitudes}]{\text{genus zero}}$	closed strings on Calabi-Yau 4-folds

- Mathematically: proposed relations between Gromov-Witten theories

Open/Closed Correspondence

- Mathematically: proposed relations between Gromov-Witten theories

open GW on toric CY 3-folds relative to Lagrangians $\xleftrightarrow{g=0}$ closed GW on toric CY 4-folds

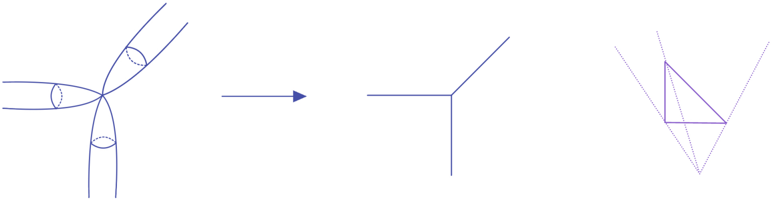
- Should hold on multiple levels:
  - Numerical invariants at individual curve classes
  - Generating functions
  - Givental-style mirror symmetry (J- and I-functions)
  - B-model mirror families, periods, Picard-Fuchs systems
  - Wall-crossings, crepant transformations
  - ...

Road Map of Construction

Open $\longrightarrow$	Relative $\longrightarrow$	Local $\longrightarrow$	Closed
(X, L)	$(X \sqcup D, D)$	$\mathcal{O}_{X \sqcup D}(-D)$	$\widetilde{X}$
$\dim_{\mathbb{C}} = 3$	3	4	4

Open Geometry

- X: $\dim_{\mathbb{C}} = 3$ toric Calabi-Yau manifold (or orbifold)
    - E.g. $\mathbb{C}^3$



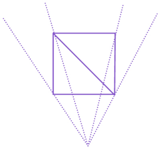
$\mathbb{C}$ coordinate axes

image under moment map of real CY 2-subtorus $T'_{\mathbb{R}}$

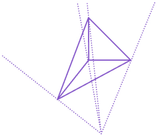
toric fan

Open Geometry

- X: $\dim_{\mathbb{C}} = 3$ toric Calabi-Yau manifold
  - Additional examples:



resolved conifold
$\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{P}^1$



$\mathcal{O}_{\mathbb{P}^2}(-3)$

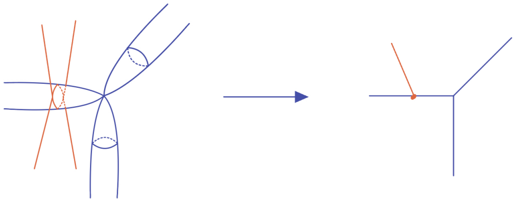
  - More general examples come from canonical bundles of toric surfaces
  - Toric fan = cone over polytope with regular triangulation

Open Geometry

- X: $\dim_{\mathbb{C}} = 3$ toric Calabi-Yau manifold
- We assume that X is semi-projective, i.e. a symplectic quotient
  - $\exists$ Hamiltonian $U(1)^k$-action on $\mathbb{C}^{k+3}$, with moment map $\mu : \mathbb{C}^{k+3} \longrightarrow \mathbb{R}^k$
  - $X = \mu^{-1}(r)/U(1)^k$, where r is a Kähler class
  - Standard Kähler form on $\mathbb{C}^{k+3}$ descends to symplectic form on X
- Equivalently: X is a GIT quotient, support of fan is convex

Aganagic-Vafa Lagrangian

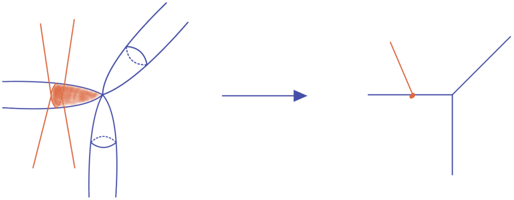
- X: $\dim_{\mathbb{C}} = 3$ toric Calabi-Yau manifold
- L: Lagrangian of Aganagic-Vafa type
  - $L = (\mu^{-1}(r) \cap \text{codim}_{\mathbb{R}} = 3 \text{ constraint})/U(1)^k$
  - Preserved under action of real CY 2-subtorus $T'_{\mathbb{R}}$



  - Topology: non-compact, $\cong S^1 \times \mathbb{R}^2$ in smooth case
  - Intersects a unique 1-dim torus orbit - we assume this orbit is non-compact

Aganagic-Vafa Lagrangian

- X: $\dim_{\mathbb{C}} = 3$ toric Calabi-Yau manifold
- L: Lagrangian of Aganagic-Vafa type
  - L bounds a disk B in the torus orbit



  - $H_1(L) = \mathbb{Z}[\partial B]$, $H_2(X, L) = H_2(X) \oplus \mathbb{Z}[B]$
- $f \in \mathbb{Z}$: additional parameter called framing of L

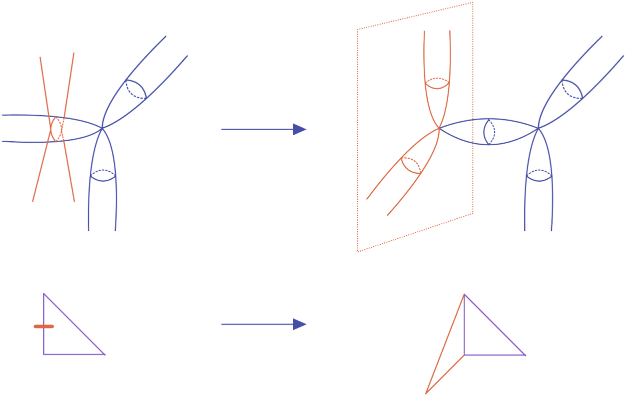
Open Gromov-Witten Invariants

- Virtual counts of stable maps from bordered Riemann surfaces to (X, L)
  - Topological vertex, all-genus mirror symmetry, crepant transformations...
  - Current project: understand them by relating to closed invariants
- We focus on disk invariants
  - Curve class: $\beta' = \beta + d[B] \in H_2(X, L)$
  - Interior insertions: $\gamma_1, \dots, \gamma_n \in H^2(X; \mathbb{Q})$
  - Defined by $T'_{\mathbb{R}}$-localization

$$\langle \gamma_1, \dots, \gamma_n \rangle_{\beta', d}^{X,(L,f)} := \int_{[\overline{\mathcal{M}}_{(0,1),n}(X,L|\beta',d)^{T'_{\mathbb{R}}}]^{\mathrm{vir}}} \frac{\iota^* \prod_{i=1}^n \mathrm{ev}_i^* \gamma_i}{e_{T'_{\mathbb{R}}}(N^{\mathrm{vir}})} \Bigg|_{\text{wt restriction}} \in \mathbb{Q}$$

From Open to Relative

- Relative geometry: add a new toric divisor $D \cong \mathbb{C}^2$ to X depending on (L, f)

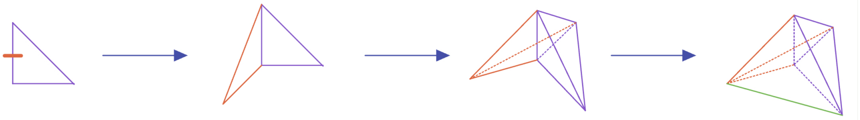


$X = \mathbb{C}^3$ $\qquad\qquad$ $X \sqcup D = \mathcal{O}(f) \oplus \mathcal{O}(-f-1)/\mathbb{P}^1$

- $(X \sqcup D, D)$ is log Calabi-Yau: $K_{X \sqcup D} = \mathcal{O}_{X \sqcup D}(-D)$
- Isomorphism $H_2(X, L) \xrightarrow{\sim} H_2(X \sqcup D)$, $[B] \mapsto [\mathbb{P}^1]$

From Relative to Local and Closed

- Local geometry $\mathcal{O}_{X\sqcup D}(-D)$: $\dim_{\mathbb{C}} = 4$ toric Calabi-Yau manifold
- Closed geometry $\widetilde{X}$: semi-projective partial compactification of $\mathcal{O}_{X\sqcup D}(-D)$



$\mathbb{C}^3$ $\quad$ $\mathcal{O}(f)\oplus\mathcal{O}(-f-1)/\mathbb{P}^1$ $\quad$ $\mathcal{O}(f)\oplus\mathcal{O}(-f-1)\oplus\mathcal{O}(-1)/\mathbb{P}^1$ $\quad$ $\mathcal{O}(-f-1)\oplus\mathcal{O}(-1)/\mathbb{P}(1,1,f)$ $(f>0)$

- Inclusion $\iota : X \longrightarrow X \sqcup D \longrightarrow \mathcal{O}_{X\sqcup D}(-D) \longrightarrow \widetilde{X}$
  - Curve classes: $\iota_* : H_2(X, L) \longrightarrow H_2(\widetilde{X})$
  - Insertions: $\iota^* : H^2(\widetilde{X};\mathbb{Q}) \longrightarrow H^2(X;\mathbb{Q})$

Closed Gromov-Witten Invariants

- Virtual counts of stable maps from (borderless) Riemann surfaces to $\widetilde{X}$
  - Curve class: $\widetilde{\beta} \in H_2(\widetilde{X})$
  - Insertions: $\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n \in H^2(\widetilde{X}; \mathbb{Q})$
  - Additional fixed insertion $\widetilde{\gamma} \in H^4_{\widetilde{T}'}(\widetilde{X}; \mathbb{Q})$ supported on fiber over D
  - Defined by localization using complex CY 3-subtorus $\widetilde{T}'$

$$\langle \widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n, \widetilde{\gamma} \rangle_{\widetilde{\beta}}^{\widetilde{X}, f} := \int_{[\overline{\mathcal{M}}_{0,n+1}(\widetilde{X}, \widetilde{\beta})^{\widetilde{T}'}]^{\mathrm{vir}}} \frac{\iota^* \prod_{i=1}^n \mathrm{ev}_i^* \widetilde{\gamma}_i \cdot \mathrm{ev}_{n+1}^* \widetilde{\gamma}}{e_{\widetilde{T}'}(N^{\mathrm{vir}})} \Big|_{\text{wt restriction}} \in \mathbb{Q}$$

Numerical Open/Closed Correspondence

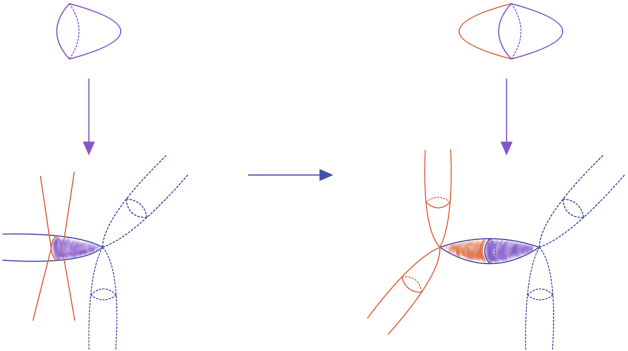
- Take any $\beta' = \beta + d[B] \in H_2(X, L) \quad \Rightarrow \quad \widetilde{\beta} = \iota_*(\beta') \in H_2(\widetilde{X})$
- $\gamma_1, \ldots, \gamma_n \in H^2(X; \mathbb{Q}) \quad \Rightarrow \quad$ lifts $\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_n \in H^2(\widetilde{X}; \mathbb{Q})$

Thm (Liu-Y)

We have

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{\beta', d}^{X,(L,f)} = \langle \widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_n, \widetilde{\gamma} \rangle_{\widetilde{\beta}}^{\widetilde{X}, f}$$

Numerical Correspondence - Proof by Picture



- Picture gives injective map

$$\left\{\begin{matrix}\text{components of fixed locus of} \\ \text{moduli of open stable maps}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{components of fixed locus of} \\ \text{moduli of closed stable maps}\end{matrix}\right\}$$

- Show that additional components on RHS don't contribute

Numerical Open/Closed Correspondence

Thm (Liu-Y)

We have

$$\langle \gamma_1, \dots, \gamma_n \rangle^{X,(L,f)}_{\beta',d} = \langle \widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n, \widetilde{\gamma} \rangle^{\widetilde{X},f}_{\widetilde{\beta}}$$

- Both sides are related to relative invariants of $(X \sqcup D, D)$
- Open/relative: already known [Li-Liu-Liu-Zhou, Fang-Liu]
  - Originates from mathematical theory of topological vertex
  - Holds for all genera and boundary winding profiles
- Relative/closed: instance of log-local principle [van Garrel-Graber-Ruddat]
  - General class of non-compact examples
  - Generalizes examples of [Bousseau-Brini-van Garrel] from Looijenga pairs

Numerical Open/Closed Correspondence

Thm (Liu-Y)

We have

$$\langle \gamma_1, \ldots, \gamma_n \rangle^{X,(L,f)}_{\beta',d} = \langle \widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_n, \widetilde{\gamma} \rangle^{\widetilde{X},f}_{\widetilde{\beta}}$$

- Potential applications (for future study): structures in open Gromov-Witten theory
  - Open WDVV
  - Open/closed Gopakumar-Vafa invariants
  - ...

Next

- Levels of open/closed correspondence:
  - Numerical invariants at individual curve classes ✓
  - Generating functions ←
  - Givental-style mirror symmetry (J- and I-functions) ←
  - B-model mirror families, periods, Picard-Fuchs systems
  - Wall-crossings, crepant transformations

Generating Functions

- Setup
  - Take basis $u_1, \ldots, u_k \in H^2(X;\mathbb{Q}) \quad \Rightarrow \quad$ lifts $\widetilde{u}_1, \ldots, \widetilde{u}_k \in H^2(\widetilde{X};\mathbb{Q})$
  - Take $\widetilde{u}_{k+1} \in H^2(\widetilde{X};\mathbb{Q})$ as class of toric divisor corresponding to D

    $\Rightarrow$ completes $\widetilde{u}_a$'s into basis
  - Set $\boldsymbol{\tau}_2 = \tau_1 u_1 + \cdots + \tau_k u_k$, $\widetilde{\boldsymbol{\tau}}_2 = \widetilde{\tau}_1 \widetilde{u}_1 + \cdots + \widetilde{\tau}_{k+1} \widetilde{u}_{k+1}$
  - t: additional variable for open sector
- Generating function of disk invariants:

$$F^{X,L,f}(\boldsymbol{\tau}_2, t) := \sum_{\beta' = \beta + d[B]} \sum_{n \in \mathbb{Z}_{\geq 0}} \frac{\langle \boldsymbol{\tau}_2^n \rangle^{X,(L,f)}_{\beta',d}}{n!} t^d$$

- Generating function of closed invariants:

$$\langle\!\langle \widetilde{\gamma} \rangle\!\rangle^{\widetilde{X},f}(\widetilde{\boldsymbol{\tau}}_2) := \sum_{\widetilde{\beta}} \sum_{n \in \mathbb{Z}_{\geq 0}} \frac{\langle \widetilde{\boldsymbol{\tau}}_2^n, \widetilde{\gamma} \rangle^{\widetilde{X},f}_{\widetilde{\beta}}}{n!}$$

Correspondence of Generating Functions

- Generating function of disk invariants:

$$F^{X,L,f}(\boldsymbol{\tau}_2, t) := \sum_{\beta'=\beta+d[B]} \sum_{n\in\mathbb{Z}_{\geq 0}} \frac{\langle \boldsymbol{\tau}_2^n \rangle^{X,(L,f)}_{\beta',d}}{n!} t^d$$

- Generating function of closed invariants:

$$\langle\langle \widetilde{\gamma} \rangle\rangle^{\widetilde{X},f}(\widetilde{\boldsymbol{\tau}}_2) := \sum_{\widetilde{\beta}} \sum_{n\in\mathbb{Z}_{\geq 0}} \frac{\langle \widetilde{\boldsymbol{\tau}}_2^n, \widetilde{\gamma} \rangle^{\widetilde{X},f}_{\widetilde{\beta}}}{n!}$$

Thm (Liu-Y)

We have

$$F^{X,L,f}(\boldsymbol{\tau}_2, t) = \langle\langle \widetilde{\gamma} \rangle\rangle^{\widetilde{X},f}(\widetilde{\boldsymbol{\tau}}_2)$$

under $\widetilde{\tau}_a = \tau_a$ for $a = 1, \dots, k$ and $\widetilde{\tau}_{k+1} = \log t$.

Correspondence of Generating Functions

Thm (Liu-Y)

We have

$$F^{X,L,f}(\boldsymbol{\tau}_2, t) = \langle\langle \widetilde{\gamma} \rangle\rangle^{\widetilde{X},f}(\widetilde{\boldsymbol{\tau}}_2)$$

under $\widetilde{\tau}_a = \tau_a$ for $a = 1, \dots, k$ and $\widetilde{\tau}_{k+1} = \log t$.

- From Gromov-Witten theory/quantum cohomology:

$$\langle\langle \widetilde{\gamma} \rangle\rangle^{\widetilde{X},f} \quad \text{``} = \text{''} \quad [z^{-2}]\left(1, S_{\widetilde{X}}^{\widetilde{T}'}(z)\widetilde{\gamma}\right)_{\widetilde{X}}^{\widetilde{T}'} \quad = \quad [z^{-2}]\left(J_{\widetilde{X}}^{\widetilde{T}'}(z), \widetilde{\gamma}\right)_{\widetilde{X}}^{\widetilde{T}'}$$

  - $S_{\widetilde{X}}^{\widetilde{T}'}$: fundamental solution to $\widetilde{T}'$-equivariant QDE of $\widetilde{X}$
  - $J_{\widetilde{X}}^{\widetilde{T}'}$: $\widetilde{T}'$-equivariant J-function of $\widetilde{X}$
  - $[z^{-2}]$: taking coefficient of z^{-2}

Thm (Liu-Y)

We have

$$F^{X,L,f}(\boldsymbol{\tau}_2, t) = [z^{-2}]\left(J_{\widetilde{X}}^{\widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2, z), \widetilde{\gamma}\right)_{\widetilde{X}}^{\widetilde{T}'}\Bigg|_{\text{wt restriction}}$$

Compatibility with Mirror Symmetry

$$
\begin{array}{ccc}
F^{X,L,f} & \longleftrightarrow & W^{X,L,f} \\
\uparrow & & \uparrow \\
J_{\widetilde{X}}^{\widetilde{T}'} & \longleftrightarrow & I_{\widetilde{X}}^{\widetilde{T}'}
\end{array}
$$

- Left: A-model open/closed correspondence

Thm (Liu-Y)

We have

$$F^{X,L,f}(\boldsymbol{\tau}_2, t) = [z^{-2}] \left(J_{\widetilde{X}}^{\widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2, z), \widetilde{\gamma} \right)_{\widetilde{X}}^{\widetilde{T}'} \Bigg|_{\text{wt restriction}}$$

Compatibility with Mirror Symmetry

$$\begin{array}{ccc} F^{X,L,f} & \longleftrightarrow & W^{X,L,f} \\ \uparrow & & \uparrow \\ J_{\widetilde{X}}^{\widetilde{T}'} & \longleftrightarrow & I_{\widetilde{X}}^{\widetilde{T}'} \end{array}$$

- Bottom: Toric mirror theorem [Givental, Coates-Corti-Iritani-Tseng]

$$J_{\widetilde{X}}^{\widetilde{T}'}(\widetilde{\boldsymbol{\tau}}_2, z) = I_{\widetilde{X}}^{\widetilde{T}'}(\widetilde{q}, z)$$

  under closed mirror map $\widetilde{\boldsymbol{\tau}}_2 = \widetilde{\boldsymbol{\tau}}_2(\widetilde{q})$

- $I_{\widetilde{X}}^{\widetilde{T}'}$: explicit hypergeometric function in B-model variables $\widetilde{q} = (\widetilde{q}_1, \dots, \widetilde{q}_{k+1})$

Compatibility with Mirror Symmetry

$$\begin{array}{ccc} F^{X,L,f} & \longleftrightarrow & W^{X,L,f} \\ \uparrow & & \uparrow \\ J_{\widetilde{X}}^{\widetilde{T}'} & \longleftrightarrow & I_{\widetilde{X}}^{\widetilde{T}'} \end{array}$$

- Top: Open mirror theorem [Fang-Liu, Fang-Liu-Tseng]

$$F^{X,L,f}(\boldsymbol{\tau}_2, t) = W^{X,L,f}(q, x)$$

under closed mirror map $\boldsymbol{\tau}_2 = \boldsymbol{\tau}_2(q)$ and open mirror map $t = t(q, x)$

- $W^{X,L,f}(q, x)$: explicit hypergeometric function in B-model closed variables $q = (q_1, \ldots, q_k)$ and open variable x

Compatibility with Mirror Symmetry

$$\begin{array}{ccc} F^{X,L,f} & \longleftrightarrow & W^{X,L,f} \\ \uparrow & & \uparrow \\ J_{\widetilde{X}}^{\widetilde{T}'} & \longleftrightarrow & I_{\widetilde{X}}^{\widetilde{T}'} \end{array}$$

- Right: B-model open/closed correspondence

Thm (Liu-Y)

We have

$$W^{X,L,f}(q,x) = [z^{-2}]\left(I_{\widetilde{X}}^{\widetilde{T}'}(\widetilde{q},z),\widetilde{\gamma}\right)_{\widetilde{X}}^{\widetilde{T}'}\Bigg|_{\text{wt restriction}}$$

under $\widetilde{q}_a = q_a$ for $a = 1,\dots,k$ and $\widetilde{q}_{k+1} = x$.

Compatibility with Mirror Symmetry

$$
\begin{array}{ccc}
F^{X,L,f} & \longleftrightarrow & W^{X,L,f} \\
\uparrow & & \uparrow \\
J_{\widetilde{X}}^{\widetilde{T}'} & \longleftrightarrow & I_{\widetilde{X}}^{\widetilde{T}'}
\end{array}
$$

- Upshot: we establish left/right sides and verify "commutativity" of diagram
- Can recover any of top/left/right from the other two

Current/Future Developments

- Levels of open/closed correspondence:
  - Numerical invariants at individual curve classes ✓
  - Generating functions ✓
  - Givental-style mirror symmetry (J- and I-functions) ✓
  - B-model mirror families, periods, Picard-Fuchs systems ←
  - Wall-crossings, crepant transformations ←

Extended Picard-Fuchs System and Periods

- Initial observation: open mirror map and disk function give solutions to an extension of Picard-Fuchs system of DEs specified by X
- And, this extended system coincides with Picard-Fuchs system specified by $\widetilde{X}$
- Known: solutions to Picard-Fuchs can be given by periods on Hori-Vafa mirror

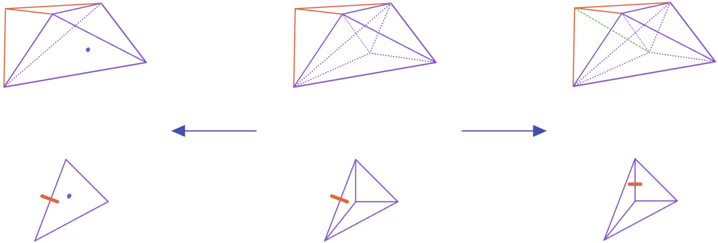
$$\int_\Gamma \Omega_q, \qquad \int_{\widetilde{\Gamma}} \widetilde{\Omega}_{\widetilde{q}}$$

  - $\Gamma \in H_3(X_q^\vee)$, $\widetilde{\Gamma} \in H_4(\widetilde{X}_{\widetilde{q}}^\vee)$
  - $\Omega, \widetilde{\Omega}$ holomorphic volume forms
- Work in progress: we find divisor $Y_{q,x} \subset X_q^\vee$ such that
  - Periods over $\Gamma \in H_3(X_q^\vee, Y_{q,x})$ recover open mirror map and disk function
  - $\exists$ isomorphism $\alpha : H_3(X_q^\vee, Y_{q,x}) \xrightarrow{\sim} H_4(\widetilde{X}_{\widetilde{q}}^\vee)$ such that

$$\int_\Gamma \Omega_q = \frac{1}{2\pi\sqrt{-1}} \int_{\alpha(\Gamma)} \widetilde{\Omega}_{\widetilde{q}}$$

Wall-Crossings and Crepant Transformations

open or closed phase shifts on (X, L) $\longleftrightarrow$ closed phase shifts on $\widetilde{X}$



closed phase shift (crepant resolution)

open phase shift (outer to inner)

Thank you!